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On the Sequential Composition of the Moore-Penrose Matrix Inverse

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1 Some Preliminary Results

Definition 1 (The Correction Function). *Define the correction function as the following lambda abstraction:*

$$\lambda x, y, u_1, u_2, v_1, v_2.((x^\dagger\ u_2\ u_1), (v_2\ v_1\ y^\dagger))$$

From now on, we refer to this function as:

$$\delta(x, y) = \lambda u_1, u_2, v_1, v_2.((x^\dagger\ u_2\ u_1), (v_2\ v_1\ y^\dagger))$$

Based on this correction function, we define an abstraction triple as follows:

Definition 2 (An Abstraction Triple). *Given a Moore-Penrose abstraction pair, $(\alpha, \alpha^\dagger)$, an abstraction triple is defined as the triple, $(\alpha, \alpha^\dagger, \delta(\alpha, \alpha))$.*

Now, define the *corrected sequential composition* of two correction abstraction triples as follows.

Definition 3 (Corrected Sequential Composition). *Given two corrected abstraction triples, $(\alpha, \alpha^\dagger, \delta(\alpha, \alpha))$ and $(\beta, \beta^\dagger, \delta(\beta, \beta))$, then the corrected sequential composition, $\dotplus$, is defined as follows:*

$$(\alpha, \alpha^\dagger, \delta(\alpha, \alpha))\ \dotplus\ (\beta, \beta^\dagger, \delta(\beta, \beta)) = (\alpha\beta, \beta^\dagger\alpha^\dagger, \delta(\alpha, \beta))$$

Such that the following conditions apply,

1- $(\alpha\beta)(\delta(\alpha, \beta)(\beta^\dagger\alpha^\dagger\beta^\dagger\alpha^\dagger)) = \pi_{\beta^\dagger\alpha^\dagger}$ *and,*
2- $(\beta^\dagger\alpha^\dagger)(\delta(\alpha, \beta)(\alpha\beta\alpha\beta)) = \pi_{\alpha\beta}$

In fact, it is straightforward to demonstrate that

$$(\alpha, \alpha^\dagger, \delta(\alpha, \alpha))\ \dotplus\ (I, I^\dagger, \delta(I, I)) = (\alpha I, I^\dagger\alpha^\dagger, \delta(\alpha, I))$$

has the property,

$$\delta(\alpha, I)(I^\dagger\alpha^\dagger I^\dagger\alpha^\dagger) = \alpha^\dagger$$

and symmetrically,

$$\delta(\alpha, I)(\alpha I \alpha I) = \alpha$$

This promotes the philosophy that an abstraction triple is a more general concept compared to the Moore-Penrose pair. This is specially shown by defining the set of *sequentially-composed* abstraction triples:

$$\mathcal{S} = \{(v, v^\dagger, \delta(v, v)) \dotplus (v', v'^\dagger, \delta(v', v')) \mid v, v' \in \mathcal{V}\}$$

where $\mathcal{V}$ is the set of all abstractions including the identity abstraction, I, and $\delta(x, y)$ is the correction function. Then, it is possible to use $\mathcal{S}$ as the domain of meaning for Moore-Penrose pairs. This is done by first defining a translation function, $\mathcal{T}$, as follows:

$$\mathcal{T}[\![(\alpha, \alpha^\dagger)]\!] = (\alpha, \alpha^\dagger, \delta(\alpha, \alpha)) \ \dotplus \ (I, I^\dagger, \delta(I, I)) \tag{1}$$

$$\mathcal{T}[\![(\alpha\beta, \beta^\dagger\alpha^\dagger)]\!] = (\alpha, \alpha^\dagger, \delta(\alpha, \alpha)) \ \dotplus \ (\beta, \beta^\dagger, \delta(\beta, \beta)) \tag{2}$$

And then working over the new set, $\mathcal{S}$. Note now that the smallest abstraction unit in $\mathcal{S}$ is the corrected sequential composition of at least two abstraction triples.

For example, let us consider the composition of the translation of two Moore-Penrose pairs, $\mathcal{T}[\![(\alpha, \alpha^\dagger)]\!] \dotplus \mathcal{T}[\![(\beta, \beta^\dagger)]\!]$, in the set of $\mathcal{S}$. First we have that:

$\mathcal{T}[\![(\alpha, \alpha^\dagger)]\!] = (\alpha, \alpha^\dagger, \delta(\alpha, \alpha)) \ \dotplus \ (I, I^\dagger, \delta(I, I))$, and
$\mathcal{T}[\![(\beta, \beta^\dagger)]\!] = (\beta, \beta^\dagger, \delta(\beta, \beta)) \ \dotplus \ (I, I^\dagger, \delta(I, I))$

From which we obtain,

$$\begin{aligned}
&((\alpha, \alpha^\dagger, \delta(\alpha, \alpha)) \dotplus (I, I^\dagger, \delta(I, I))) \quad \dotplus \quad ((\beta, \beta^\dagger, \delta(\beta, \beta)) \dotplus (I, I^\dagger, \delta(I, I))) \quad = \\
&(\alpha I, I^\dagger\alpha^\dagger, \delta(\alpha, I)) \quad \dotplus \quad (\beta I, I^\dagger\beta^\dagger, \delta(\beta, I)) \quad = \\
&(\alpha I \beta I, I^\dagger\beta^\dagger I^\dagger\alpha^\dagger, \delta(\alpha I, \beta I)) \quad = \\
&(\alpha\beta, \beta^\dagger\alpha^\dagger, \delta(\alpha, \beta)) \quad = \\
&(\alpha, \alpha^\dagger, \delta(\alpha, \alpha)) \ \dotplus \ (\beta, \beta^\dagger, \delta(\beta, \beta)) \quad = \\
&\mathcal{T}[\![(\alpha\beta, \beta^\dagger\alpha^\dagger)]\!]
\end{aligned}$$

Furthermore, it is possible to demonstrate that:

$$\begin{aligned}
&((\alpha\beta, \beta^\dagger\alpha^\dagger, \delta(\alpha, \beta)) \quad \dotplus \quad (\gamma\zeta, \zeta^\dagger\gamma^\dagger, \delta(\gamma, \zeta))) \quad = \\
&((\alpha\beta\gamma\zeta, \zeta^\dagger\gamma^\dagger\beta^\dagger\alpha^\dagger, \delta(\alpha\beta, \gamma\zeta))
\end{aligned}$$

This promotes compositionality!